

GENERALIZED QUANTUM MONTGOMERY IDENTITY AND OSTROWSKI TYPE INEQUALITIES FOR PREINNVEX FUNCTIONS

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ABSTRACT. In this research, we give a generalized version of the quantum Montgomery identity using the quantum integral. We establish some new inequalities of Ostrowski type by means of newly derived identity. Moreover, we consider the special cases of the newly obtained results and prove several new and known Ostrowski and midpoint inequalities.

Keywords: Ostrowski inequalities, midpoint inequalities, q -integral, quantum calculus, preinvex functions.

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1. INTRODUCTION

Various types of integral inequalities have attracted the attention of several mathematicians over the last decades. These inequalities play a significant role in the study of various classes of equations such as integro-differential equations and impulse differential equations. That is why a vast amount of research activities are being carried out on this subject.

The following is a classical integral inequality associated with the differentiable mappings (see, [36]):

Theorem 1.1. *If the mapping $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on (a, b) and integrable on $[a, b]$, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_{\infty},$$

for all $x \in [a, b]$, where $\|f'\|_{\infty} = \sup_{t \in (a,b)} |f'(t)| < +\infty$. Moreover, $\frac{1}{4}$ is the best possible constant.

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Theorem 1.2. [15] *Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable on (a, b) and integrable on $[a, b]$. If $|f'(\varkappa)| \leq M$, for every $\varkappa \in [a, b]$, then the following inequality*

$$\left| f(\varkappa) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{M}{b-a} \left[\frac{(\varkappa - a)^2 + (b - \varkappa)^2}{2} \right], \quad (1)$$

holds.

Following is the well-known Montgomery identity:

Lemma 1.1. [29] *If the mapping $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on (a, b) and integrable on $[a, b]$, then following identity holds:*

$$f(\varkappa) - \frac{1}{b-a} \int_a^b f(t) dt = \int_a^{\varkappa} \frac{t-a}{b-a} f'(t) dt + \int_{\varkappa}^b \frac{t-b}{b-a} f'(t) dt. \quad (2)$$

By changing the variables, we can rewrite (2) in the following way:

Lemma 1.2. [37, Lemma 1] *Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on (a, b) and integrable on $[a, b]$. Then the following equality holds:*

$$\begin{aligned} f(\varkappa) - \frac{1}{b-a} \int_a^b f(t) dt &= (b-a) \left[\int_0^{\frac{b-\varkappa}{b-a}} t f'(ta + (1-t)b) dt \right. \\ &\quad \left. + \int_{\frac{b-\varkappa}{b-a}}^1 (t-1) f'(ta + (1-t)b) dt \right]. \end{aligned} \quad (3)$$

On the other hand, quantum calculus (shortly, q -calculus) deals with the concept of calculus without limits, where the classical mathematical results are obtained by taking the limit as $q \rightarrow 1^-$.

The study of q -calculus was initiated in the early 20th century after the work of Jackson (1910) who defined an integral later known as the q -Jackson integral (see, [9, 17, 18, 21, 23]). In q -calculus, the classical derivative is replaced by the q -difference operator to deal with non-differentiable functions. For more discussion on this subject, we refer to [3, 13]. Applications of q -calculus can be found in various disciplines of mathematics and physics (see, [7, 20, 38, 43]).

Many well-known integral inequalities such as Hölder's inequality, Hermite-Hadamard inequality, Ostrowski's inequality, Simpson's inequality, Newton's inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss-Cebysev and other integral inequalities in classical analysis have been proved and applied in the setup of q -calculus using the classical concept of convexity. For more results in this direction, we refer to [1, 2, 5, 6, 10, 11, 12, 19, 22, 24, 26, 28, 30, 31, 32, 34, 35, 40, 42, 44, 45].

The purpose of this paper is to study Ostrowski's inequalities for q -differentiable preinvex functions by applying the newly defined concept of q^b -integral. We also discuss the relation of the results obtained herein with comparable results in the existing literature.

The organization of this paper is as follows: In Section 2, we summarize the concept of q -calculus and some related work in this setup is given. In Section 3, the proof of Montgomery identity for q^b -integral is given. Using the Montgomery identity for q^b -integral, Ostrowski's type inequalities are obtained. Some special cases of our main results are presented in Section 4. The relation of the obtained results with the comparable results in the existing literature is also discussed. Section 5 contains some conclusions and further directions for future research. We believe that the study initiated in this paper may provide a good source of inspiration to the researchers working on integral inequalities and their applications.

2. PRELIMINARIES OF q -CALCULUS AND SOME INEQUALITIES

The basic notions and findings which are needed in the sequel to prove our crucial results are reviewed in this section. Throughout this paper, we assume that $a < b$ and $0 < q < 1$. Let ω be a nonempty closed set in \mathbb{R}^n , $f : \omega \rightarrow \mathbb{R}$ be a continuous function and $\eta(.,.) : \omega \times \omega \rightarrow \mathbb{R}^n$ be a continuous bifunction.

Definition 2.1. [14] A set ω is said to be an invex set with respect to bifunction $\eta(.,.)$ if

$$b + t\eta(a, b) \in \omega, \forall a, b \in \omega, t \in [0, 1].$$

The invex set ω is also known as η -connected set.

Definition 2.2. [14] A mapping f is said to be preinvex with respect to an arbitrary bifunction $\eta(.,.)$ if the following inequality holds:

$$f(b + t\eta(a, b)) \leq tf(a) + (1 - t)f(b), \forall a, b \in \omega, t \in [0, 1].$$

The function f is called preconcave if $-f$ is preinvex.

Various authors used this definition and offered different kinds of integral inequalities. For some recent results one can read [14, 16, 27].

Remark 2.1. If we set $\eta(a, b) = a - b$, then the definition of preinvex functions reduces to the definition of convex functions given below;

$$f(b + t(a - b)) \leq tf(a) + (1 - t)f(b), \forall a, b \in \omega, t \in [0, 1].$$

Now, we present some well-known concepts and theorems for q -derivative and q -integral of a function f on $[a, b]$. We set the following notation(see, [23]):

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}, \quad q \in (0, 1).$$

Definition 2.3. [5, 23] For a continuous function $f : [a, b] \rightarrow \mathbb{R}$, the q_a -derivative of f at $x \in [a, b]$ is characterized by the expression

$${}_a D_q f(x) = \frac{f(x) - f(qx + (1 - q)a)}{(1 - q)(x - a)}, \quad x \neq a. \quad (4)$$

If $x = a$, we define ${}_a D_q f(a) = \lim_{x \rightarrow a} {}_a D_q f(x)$ if it exists and it is finite.

Definition 2.4. [41] Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, the q_a -definite integral on $[a, b]$ is defined by

$$\int_a^x f(s) {}_a d_q s = (1 - q)(x - a) \sum_{n=0}^{\infty} q^n f(q^n x + (1 - q^n)a), \quad x \in [a, b]. \quad (5)$$

Remark 2.2. If $a = 0$ in (5), then $\int_0^x f(s) {}_0 d_q s = \int_0^x f(s) d_q s$, where $\int_0^x f(s) d_q s$ is the familiar q -definite integral (see, [23]) on $[0, x]$ defined by

$$\int_0^x f(s) {}_0 d_q s = \int_0^x f(s) d_q s = (1 - q)x \sum_{n=0}^{\infty} q^n f(q^n x). \quad (6)$$

Definition 2.5. If $c \in (a, x)$, then the q_a -definite integral on $[c, x]$ is expressed as

$$\int_c^x f(s) {}_a d_q s = \int_a^x f(s) {}_a d_q s - \int_a^c f(s) {}_a d_q s. \quad (7)$$

Alp et al. [5], proved the following q -Hermite-Hadamard inequality:

Theorem 2.1. (*q-Hermite-Hadamard inequality*) Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex differentiable function on $[a, b]$ and $0 < q < 1$. Then, we have

$$f\left(\frac{qa + b}{[2]_q}\right) \leq \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \leq \frac{qf(a) + f(b)}{[2]_q}.$$

On the other hand, Bermudo et al. [8], gave the following new definitions of quantum integral and derivative. In the same paper, the authors proved a new variant of the quantum Hermite-Hadamard type inequality linked with their newly defined quantum integral:

Definition 2.6. [8] Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, the q^b -definite integral on $[a, b]$ is given by

$$\int_a^b f(x) {}^b d_q x = (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(q^n a + (1 - q^n)b) = (b - a) \int_0^1 f(sa + (1 - s)b) d_q s.$$

Definition 2.7. [8] Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. The q^b -derivative of f at $x \in [a, b]$ is given by

$${}^b D_q f(x) = \frac{f(qx + (1 - q)b) - f(x)}{(1 - q)(b - x)}, \quad x \neq b.$$

If $x = b$, we define ${}^b D_q f(b) = \lim_{x \rightarrow b} {}^b D_q f(x)$ if it exists and it is finite.

Theorem 2.2. [8] If $f : [a, b] \rightarrow \mathbb{R}$ is a convex differentiable function on $[a, b]$ and $0 < q < 1$. Then, *q-Hermite-Hadamard inequalities* are given as follows:

$$f\left(\frac{a + qb}{[2]_q}\right) \leq \frac{1}{b - a} \int_a^b f(x) {}^b d_q x \leq \frac{f(a) + qf(b)}{[2]_q}. \tag{8}$$

Lemma 2.1. [5] For $\alpha \in \mathbb{R} \setminus \{-1\}$, the following formula holds:

$$\int_a^x (s - a)^\alpha {}_a d_q s = \frac{(x - a)^{\alpha+1}}{[\alpha + 1]_q}. \tag{9}$$

3. MONTGOMERY IDENTITY AND OSTROWSKI TYPE INEQUALITIES FOR QUANTUM INTEGRALS

In this section, we first prove the generalized version of the quantum Montgomery identity for q^b -definite integrals. Then, by using the established identity, we establish some Ostrowski type inequalities.

Let us start with the following useful lemma, which is a generalized Montgomery identity for q^b -integral.

Lemma 3.1 (Quantum Montgomery Identity). Let $f : I = [b + \eta(a, b), b] \rightarrow \mathbb{R}$ be a q -differentiable function such that ${}^b D_q f$ is q -integrable on I° , then the following identity holds:

$$\frac{1}{\eta(b, a)} \int_{b+\eta(a,b)}^b f(t) {}^b d_q t - f(x) = \eta(b, a) \int_0^1 \Psi_q(t) {}^b D_q f(b + t\eta(a, b)) d_q t \tag{10}$$

where

$$\Psi_q(t) = \begin{cases} qt; & \text{if } t \in [0, \wp_\eta], \\ qt - 1; & \text{if } t \in [\wp_\eta, 1], \end{cases}$$

and $\wp_\eta = \frac{b-\varkappa}{\eta(b,a)}$.

Proof. Using the fundamental properties of quantum integrals and from Definitions 2 and 2, we obtain that

$$\begin{aligned} & \int_0^1 \Psi_q(t) {}^b D_q f(b + t\eta(a, b)) d_q t \\ &= \int_0^{\wp_\eta} qt {}^b D_q f(b + t\eta(a, b)) d_q t + \int_{\wp_\eta}^1 (qt - 1) {}^b D_q f(b + t\eta(a, b)) d_q t \\ &= \int_0^{\wp_\eta} qt {}^b D_q f(b + t\eta(a, b)) d_q t + \int_0^1 (qt - 1) {}^b D_q f(b + t\eta(a, b)) d_q t \\ &\quad - \int_0^{\wp_\eta} (qt - 1) {}^b D_q f(b + t\eta(a, b)) d_q t \\ &= \int_0^1 (qt - 1) {}^b D_q f(b + t\eta(a, b)) d_q t + \int_0^{\wp_\eta} {}^b D_q f(b + t\eta(a, b)) d_q t \\ &= \int_0^1 qt {}^b D_q f(b + t\eta(a, b)) d_q t - \int_0^1 {}^b D_q f(b + t\eta(a, b)) d_q t \\ &\quad + \int_0^{\wp_\eta} {}^b D_q f(b + t\eta(a, b)) d_q t \\ &= \int_0^1 qt \frac{f(b + qt\eta(a, b)) - f(b + t\eta(a, b))}{t(1-q)\eta(b, a)} d_q t \\ &\quad - \int_0^1 \frac{f(b + qt\eta(a, b)) - f(b + t\eta(a, b))}{t(1-q)\eta(b, a)} d_q t \\ &\quad + \int_0^{\wp_\eta} \frac{f(b + qt\eta(a, b)) - f(b + t\eta(a, b))}{t(1-q)\eta(b, a)} d_q t \\ &= \frac{1}{(1-q)\eta(b, a)} \left[(1-q) \sum_{n=0}^{\infty} q^{n+1} f(b + q^{n+1}\eta(a, b)) - (1-q) \sum_{n=0}^{\infty} q^{n+1} f(b + q^n\eta(a, b)) \right] \\ &\quad - \frac{1}{(1-q)\eta(b, a)} \left[(1-q) \sum_{n=0}^{\infty} f(b + q^{n+1}\eta(a, b)) - (1-q) \sum_{n=0}^{\infty} q^n f(b + q^n\eta(a, b)) \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{(1-q)\eta(b,a)} \left[(1-q) \sum_{n=0}^{\infty} f(b+q^{n+1} \wp_{\eta} \eta(a,b)) - (1-q) \sum_{n=0}^{\infty} f(b+q^n \wp_{\eta} \eta(a,b)) \right] \\
 = & \frac{1}{\eta(b,a)} \left[\sum_{n=0}^{\infty} q^{n+1} f(b+q^{n+1} \eta(a,b)) - \sum_{n=0}^{\infty} q^{n+1} f(b+q^n \eta(a,b)) \right. \\
 & \left. + f(b+\eta(a,b)) - f(b+\eta(a,b)) \right] \\
 & - \frac{1}{\eta(b,a)} [f(b) - f(b+\eta(a,b))] + \frac{1}{\eta(b,a)} [f(b) - f(b+\wp_{\eta}\eta(a,b))] \\
 = & \frac{1}{\eta(b,a)} \left[(1-q) \sum_{n=0}^{\infty} q^n f(b+q^n \eta(a,b)) - f(b+\eta(a,b)) \right] \\
 & - \frac{1}{\eta(b,a)} [-f(b+\eta(a,b)) + f(b+\wp_{\eta}\eta(a,b))] \\
 = & \frac{1}{(\eta(b,a))^2} \int_{b+\eta(a,b)}^b f(\varkappa) {}^b d_q t + \frac{1}{\eta(b,a)} [-f(b+\eta(a,b)) + f(b+\eta(a,b)) \\
 & - f(b+\wp_{\eta}\eta(a,b))] \\
 = & \frac{1}{(\eta(b,a))^2} \int_{b+\eta(a,b)}^b f(\varkappa) {}^b d_q t - \frac{1}{\eta(b,a)} f(\varkappa).
 \end{aligned}$$

Finally, by multiplying the last identity with $\eta(b,a)$, we obtain the desired identity. Thus, the proof is completed. \square

Theorem 3.1. *We assume that the conditions of Lemma 3.1 hold. If $|{}^b D_q f|^{p_1}$ is a preinvex function on I , where $p_1 \geq 1$, then for $0 < q < 1$ we have the following inequality*

$$\begin{aligned}
 & \left| \frac{1}{\eta(b,a)} \int_{b+\eta(a,b)}^b f(t) {}^b d_q t - f(\varkappa) \right| \\
 \leq & \eta(b,a) \left[A_1^{1-\frac{1}{p_1}}(a,b,q,\varkappa) \left(\left| {}^b D_q f(a) \right|^{p_1} A_2(a,b,q,\varkappa) + \left| {}^b D_q f(b) \right|^{p_1} A_3(a,b,q,\varkappa) \right)^{\frac{1}{p_1}} \right. \\
 & \left. + A_4^{1-\frac{1}{p_1}}(a,b,q,\varkappa) \left(\left| {}^b D_q f(a) \right|^{p_1} A_5(a,b,q,\varkappa) + \left| {}^b D_q f(b) \right|^{p_1} A_6(a,b,q,\varkappa) \right)^{\frac{1}{p_1}} \right]
 \end{aligned}$$

where

$$\begin{aligned}
 A_1(a,b,q,\varkappa) &= \int_0^{\frac{b-\varkappa}{\eta(b,a)}} q t d_q \varkappa = \frac{q}{[2]_q} \left(\frac{b-\varkappa}{\eta(b,a)} \right)^2, \\
 A_2(a,b,q,\varkappa) &= \int_0^{\frac{b-\varkappa}{\eta(b,a)}} q t^2 d_q \varkappa = \frac{q}{[3]_q} \left(\frac{b-\varkappa}{\eta(b,a)} \right)^3,
 \end{aligned}$$

$$\begin{aligned}
A_3(a, b, q, \varkappa) &= \int_0^{\frac{b-\varkappa}{\eta(b,a)}} q t d_q \varkappa - \int_0^{\frac{b-\varkappa}{\eta(b,a)}} q t^2 d_q \varkappa = A_1(a, b, q, \varkappa) - A_2(a, b, q, \varkappa), \\
A_4(a, b, q, \varkappa) &= \int_{\frac{b-\varkappa}{\eta(b,a)}}^1 (1-qt) d_q t = \int_0^1 (1-qt) d_q t - \int_0^{\frac{b-\varkappa}{\eta(b,a)}} (1-qt) d_q t \\
&= \frac{1-q}{[2]_q} \left(\frac{\varkappa-a}{\eta(b,a)} \right) + \frac{q}{[2]_q} \left(\frac{\varkappa-a}{\eta(b,a)} \right)^2, \\
A_5(a, b, q, \varkappa) &= \int_{\frac{b-\varkappa}{\eta(b,a)}}^1 (t-qt^2) d_q t = \int_0^1 (t-qt^2) d_q t - \int_0^{\frac{b-\varkappa}{\eta(b,a)}} (t-qt^2) d_q t \\
&= \frac{1}{[2]_q [3]_q} - \frac{1}{[2]_q} \left(\frac{b-\varkappa}{\eta(b,a)} \right)^2 + \frac{q}{[3]_q} \left(\frac{b-\varkappa}{\eta(b,a)} \right)^3,
\end{aligned}$$

and

$$\begin{aligned}
&A_6(a, b, q, \varkappa) \\
&= \int_{\frac{b-\varkappa}{b-a}}^1 (1-t)(1-qt) d_q t \\
&= \int_0^1 (1-t)(1-qt) d_q t - \int_0^{\frac{b-\varkappa}{\eta(b,a)}} (1-t)(1-qt) d_q t \\
&= \int_0^1 (1-qt) d_q t - \int_0^1 (t-qt^2) d_q t - \int_0^{\frac{b-\varkappa}{\eta(b,a)}} (1-qt) d_q t + \int_0^{\frac{b-\varkappa}{\eta(b,a)}} (t-qt^2) d_q t \\
&= A_4(a, b, q, \varkappa) - A_5(a, b, q, \varkappa).
\end{aligned}$$

Proof. Taking the modulus in Lemma 3.1 and using the power mean inequality for quantum integrals, we obtain that

$$\left| \frac{1}{\eta(b,a)} \int_{b+\eta(a,b)}^b f(t) {}^b d_q t - f(\varkappa) \right|$$

$$\begin{aligned}
 &\leq \eta(b, a) \int_0^1 |\Psi_q(t)| \left| {}^b D_q f(b + t\eta(a, b)) \right| d_q t \\
 &\leq \eta(b, a) \left[\int_0^{\frac{b-\varkappa}{\eta(b, a)}} qt \left| {}^b D_q f(b + t\eta(a, b)) \right| d_q t \right. \\
 &\quad \left. + \int_{\frac{b-\varkappa}{\eta(b, a)}}^1 (1-qt) \left| {}^b D_q f(b + t\eta(a, b)) \right| d_q t \right] \\
 &\leq \eta(b, a) \left[\left(\int_0^{\frac{b-\varkappa}{\eta(b, a)}} qt d_q t \right)^{1-\frac{1}{p_1}} \left(\int_0^{\frac{b-\varkappa}{\eta(b, a)}} qt \left| {}^b D_q f(b + t\eta(a, b)) \right|^{p_1} d_q t \right)^{\frac{1}{p_1}} \right. \\
 &\quad \left. + \left(\int_{\frac{b-\varkappa}{\eta(b, a)}}^1 (1-qt) \right)^{1-\frac{1}{p_1}} \left(\int_{\frac{b-\varkappa}{\eta(b, a)}}^1 (1-qt) \left| {}^b D_q f(b + t\eta(a, b)) \right|^{p_1} d_q t \right)^{\frac{1}{p_1}} \right]
 \end{aligned}$$

Using the preinvexity of $|{}^b D_q f|^{p_1}$, we have

$$\begin{aligned}
 &\left| \frac{1}{\eta(b, a)} \int_{b+\eta(a, b)}^b f(t) {}^b d_q t - f(\varkappa) \right| \\
 &\leq \eta(b, a) \left[\left(\int_0^{\frac{b-\varkappa}{\eta(b, a)}} qt d_q t \right)^{1-\frac{1}{p_1}} \right. \\
 &\quad \times \left(\int_0^{\frac{b-\varkappa}{\eta(b, a)}} qt \left(\left| {}^b D_q f(a) \right|^{p_1} t + \left| {}^b D_q f(b) \right|^{p_1} (1-t) \right) d_q t \right)^{\frac{1}{p_1}} \\
 &\quad + \left(\int_{\frac{b-\varkappa}{\eta(b, a)}}^1 (1-qt) \right)^{1-\frac{1}{p_1}} \\
 &\quad \times \left(\int_{\frac{b-\varkappa}{\eta(b, a)}}^1 (1-qt) \left(\left| {}^b D_q f(a) \right|^{p_1} t + \left| {}^b D_q f(b) \right|^{p_1} (1-t) \right) d_q t \right)^{\frac{1}{p_1}} \left. \right]
 \end{aligned}$$

$$\begin{aligned}
&= \eta(b, a) \left[\left(\int_0^{\frac{b-\varkappa}{\eta(b, a)}} qt d_q t \right)^{1-\frac{1}{p_1}} \right. \\
&\quad \times \left(\left| {}^b D_q f(a) \right|^{p_1} \int_0^{\frac{b-\varkappa}{\eta(b, a)}} qt^2 d_q t + \left| {}^b D_q f(b) \right|^{p_1} \int_0^{\frac{b-\varkappa}{\eta(b, a)}} (qt - qt^2) d_q t \right)^{\frac{1}{p_1}} \\
&\quad + \left(\int_{\frac{b-\varkappa}{\eta(b, a)}}^1 (1-qt) \right)^{1-\frac{1}{p_1}} \\
&\quad \times \left(\left| {}^b D_q f(a) \right|^{p_1} \int_{\frac{b-\varkappa}{\eta(b, a)}}^1 t(1-qt) d_q t + \left| {}^b D_q f(b) \right|^{p_1} \int_{\frac{b-\varkappa}{\eta(b, a)}}^1 (1-t)(1-qt) d_q t \right)^{\frac{1}{p_1}} \Bigg] \\
&= \eta(b, a) \left[A_1^{1-\frac{1}{p_1}}(a, b, q, \varkappa) \left(\left| {}^b D_q f(a) \right|^{p_1} A_2(a, b, q, \varkappa) + \left| {}^b D_q f(b) \right|^{p_1} A_3(a, b, q, \varkappa) \right)^{\frac{1}{p_1}} \right. \\
&\quad \left. + A_4^{1-\frac{1}{p_1}}(a, b, q, \varkappa) \left(\left| {}^b D_q f(a) \right|^{p_1} A_5(a, b, q, \varkappa) + \left| {}^b D_q f(b) \right|^{p_1} A_6(a, b, q, \varkappa) \right)^{\frac{1}{p_1}} \right]
\end{aligned}$$

which completes the proof. \square

Theorem 3.2. *We assume that the conditions of Lemma 3.1 hold. Suppose $|{}^b D_q f|^{p_1}$ is a preinvex function on I , for some $p_1 > 1$ with $\frac{1}{r_1} + \frac{1}{p_1} = 1$. Then, we have*

$$\begin{aligned}
&\left| \frac{1}{\eta(b, a)} \int_{b+\eta(a, b)}^b f(t) {}^b d_q t - f(\varkappa) \right| \tag{11} \\
&\leq \eta(b, a) \left[\left(\frac{b-\varkappa}{\eta(b, a)} \right)^{1+\frac{1}{r_1}} \left(\frac{q}{[r_1+1]_q} \right)^{\frac{1}{r_1}} \right. \\
&\quad \times \left(\left| {}^b D_q f(a) \right|^{p_1} \frac{1}{[2]_q} \left(\frac{b-\varkappa}{\eta(b, a)} \right)^2 + \left| {}^b D_q f(b) \right|^{p_1} \left(\frac{b-\varkappa}{\eta(b, a)} - \frac{1}{[2]_q} \left(\frac{b-\varkappa}{\eta(b, a)} \right)^2 \right) \right)^{\frac{1}{p_1}} \\
&\quad + \left(\int_{\frac{b-\varkappa}{\eta(b, a)}}^1 (1-qt)^{r_1} d_q t \right)^{\frac{1}{r_1}} \\
&\quad \times \left(\left| {}^b D_q f(a) \right|^{p_1} \left(\frac{1}{[2]_q} \left(1 - \left(\frac{b-\varkappa}{\eta(b, a)} \right)^2 \right) \right) \right. \\
&\quad \left. + \left| {}^b D_q f(b) \right|^{p_1} \left(\frac{q}{[2]_q} - \frac{b-\varkappa}{\eta(b, a)} + \frac{1}{[2]_q} \left(\frac{b-\varkappa}{\eta(b, a)} \right)^2 \right) \right)^{\frac{1}{p_1}} \Bigg],
\end{aligned}$$

where $0 < q < 1$.

Proof. By taking the modulus in the Lemma 3.1 and using the well-known Hölder’s inequality for quantum integrals, we obtain that

$$\begin{aligned}
 & \left| \frac{1}{\eta(b, a)} \int_{b+\eta(a, b)}^b f(t) {}^b d_q t - f(\varkappa) \right| \\
 & \leq \eta(b, a) \int_0^1 |\Psi_q(t)| \left| {}^b D_q f(b + t\eta(a, b)) \right| d_q t \\
 & \leq \eta(b, a) \left[\int_0^{\frac{b-\varkappa}{\eta(b, a)}} qt \left| {}^b D_q f(b + t\eta(a, b)) \right| d_q t \right. \\
 & \quad \left. + \int_{\frac{b-\varkappa}{\eta(b, a)}}^1 (1-qt) \left| {}^b D_q f(b + t\eta(a, b)) \right| d_q t \right] \\
 & \leq \eta(b, a) \left[\left(\int_0^{\frac{b-\varkappa}{\eta(b, a)}} (qt)^{r_1} d_q t \right)^{\frac{1}{r_1}} \left(\int_0^{\frac{b-\varkappa}{\eta(b, a)}} \left| {}^b D_q f(b + t\eta(a, b)) \right|^{p_1} d_q t \right)^{\frac{1}{p_1}} \right. \\
 & \quad \left. + \left(\int_{\frac{b-\varkappa}{\eta(b, a)}}^1 (1-qt)^{r_1} d_q t \right)^{\frac{1}{r_1}} \left(\int_{\frac{b-\varkappa}{\eta(b, a)}}^1 \left| {}^b D_q f(b + t\eta(a, b)) \right|^{p_1} d_q t \right)^{\frac{1}{p_1}} \right].
 \end{aligned}$$

Using the assumption that $|{}^b D_q f|^{p_1}$ is a preinvex function, we have

$$\begin{aligned}
 & \left| \frac{1}{\eta(b, a)} \int_{b+\eta(a, b)}^b f(t) {}^b d_q t - f(\varkappa) \right| \\
 & \leq \eta(b, a) \left[\left(\int_0^{\frac{b-\varkappa}{\eta(b, a)}} (qt)^{r_1} d_q t \right)^{\frac{1}{r_1}} \right. \\
 & \quad \left. \times \left(\left| {}^b D_q f(a) \right|^{p_1} \int_0^{\frac{b-\varkappa}{\eta(b, a)}} t d_q t + \left| {}^b D_q f(b) \right|^{p_1} \int_0^{\frac{b-\varkappa}{\eta(b, a)}} (1-t) d_q t \right)^{\frac{1}{p_1}} \right]
 \end{aligned}$$

$$\begin{aligned}
& + \left(\int_{\frac{b-\varkappa}{\eta(b,a)}}^1 (1-qt)^{r_1} d_q t \right)^{\frac{1}{r_1}} \\
& \times \left[\left| {}^b D_q f(a) \right|^{p_1} \int_{\frac{b-\varkappa}{\eta(b,a)}}^1 t d_q t + \left| {}^b D_q f(b) \right|^{p_1} \int_{\frac{b-\varkappa}{\eta(b,a)}}^1 (1-t) d_q t \right]^{\frac{1}{p_1}} \\
= & \eta(b,a) \left[\left(\frac{b-\varkappa}{\eta(b,a)} \right)^{1+\frac{1}{r_1}} \left(\frac{q}{[r_1+1]_q} \right)^{\frac{1}{r_1}} \right. \\
& \times \left(\left| {}^b D_q f(a) \right|^{p_1} \frac{1}{[2]_q} \left(\frac{b-\varkappa}{\eta(b,a)} \right)^2 + \left| {}^b D_q f(b) \right|^{p_1} \left(\frac{b-\varkappa}{\eta(b,a)} - \frac{1}{[2]_q} \left(\frac{b-\varkappa}{\eta(b,a)} \right)^2 \right) \right]^{\frac{1}{p_1}} \\
& + \left(\int_{\frac{b-\varkappa}{\eta(b,a)}}^1 (1-qt)^{r_1} d_q t \right)^{\frac{1}{r_1}} \\
& \times \left(\left| {}^b D_q f(a) \right|^{p_1} \left(\frac{1}{[2]_q} \left(1 - \left(\frac{b-\varkappa}{\eta(b,a)} \right)^2 \right) \right) \right. \\
& \left. + \left| {}^b D_q f(b) \right|^{p_1} \left(\frac{q}{[2]_q} - \frac{b-\varkappa}{\eta(b,a)} + \frac{1}{[2]_q} \left(\frac{b-\varkappa}{\eta(b,a)} \right)^2 \right) \right)^{\frac{1}{p_1}} \Big]
\end{aligned}$$

which completes the proof. \square

4. SOME SPECIAL CASES

In this section, some special cases of our main results are discussed and several new results in the field of Ostrowski and midpoint type inequalities are obtained.

Remark 4.1. *If we consider Lemma 3.1, then*

(i) *On taking the limit as $q \rightarrow 1^-$ in Lemma 3.1, the identity (10) reduces to the following identity*

$$\begin{aligned}
& \frac{1}{\eta(b,a)} \int_{b+\eta(a,b)}^b f(t) dt - f(\varkappa) \\
= & \eta(b,a) \left[\int_0^{\wp_\eta} t f'(b+t\eta(a,b)) dt + \int_{\wp_\eta}^1 (t-1) f'(b+t\eta(a,b)) dt \right].
\end{aligned} \tag{12}$$

Furthermore, On taking $\eta(b,a) = -\eta(a,b) = b-a$, the identity (12) reduces to [2, Lemma 4].

(ii) If we set $\varkappa = \frac{\eta(a,b)+2b}{2}$ in (12), then we have the following identity

$$\begin{aligned} & \frac{1}{\eta(b,a)} \int_{b+\eta(a,b)}^b f(t) dt - f\left(\frac{\eta(a,b)+2b}{2}\right) \\ &= \eta(b,a) \left[\int_0^{\frac{1}{2}} t f'(b+t\eta(a,b)) dt + \int_{\frac{1}{2}}^1 (t-1) f'(b+t\eta(a,b)) dt \right] \end{aligned}$$

which was given by Noor et al. in [33, Lemma 3.10].

(iii) On taking the limit $q \rightarrow 1^-$ and $\eta(b,a) = -\eta(a,b) = b-a$ in Lemma 3.1, we obtain the Montgomery identity given in (3).

(iv) On taking $\varkappa = \frac{a+qb}{[2]_q}$ and $\eta(b,a) = -\eta(a,b) = b-a$ in Lemma 3.1, we have the following equality

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(t) {}^b d_q t - f\left(\frac{a+qb}{[2]_q}\right) \\ &= (b-a) \left[\int_0^{\frac{1}{[2]_q}} qt {}^b D_q f(ta+(1-t)b) d_q t + \int_{\frac{1}{[2]_q}}^1 (qt-1) {}^b D_q f(ta+(1-t)b) d_q t \right] \end{aligned}$$

which was given by Budak in [12].

(v) On letting $\varkappa = \frac{b+q(b+\eta(a,b))}{[2]_q}$ in Lemma 3.1, we obtain the following new identity

$$\begin{aligned} & \frac{1}{\eta(b,a)} \int_{b+\eta(a,b)}^b f(t) {}^b d_q t - f\left(\frac{b+q(b+\eta(a,b))}{[2]_q}\right) \\ &= \eta(b,a) \left[\int_0^{\frac{q}{[2]_q}} qt {}^b D_q f(b+t\eta(a,b)) d_q t + \int_{\frac{q}{[2]_q}}^1 (qt-1) {}^b D_q f(b+t\eta(a,b)) d_q t \right]. \end{aligned}$$

Remark 4.2. Let us consider Theorem 3.1. Then,

(i) By considering $p_1 = 1$ in Theorem 3.1, we have the following Ostrwoski inequality

$$\begin{aligned} & \left| \frac{1}{\eta(b,a)} \int_{b+\eta(a,b)}^b f(t) {}^b d_q t - f(\varkappa) \right| \tag{13} \\ & \leq \eta(b,a) \left[\left| {}^b D_q f(a) \right| (A_2(a,b,q,\varkappa) + A_5(a,b,q,\varkappa)) \right. \\ & \quad \left. + \left| {}^b D_q f(b) \right| (A_3(a,b,q,\varkappa) + A_6(a,b,q,\varkappa)) \right]. \end{aligned}$$

Moreover, by taking the limit as $q \rightarrow 1^-$ in (13), we have the following new Ostrowski type inequality:

$$\begin{aligned} & \left| \frac{1}{\eta(b, a)} \int_{b+\eta(a, b)}^b f(t) dt - f(\varkappa) \right| \\ & \leq \eta(b, a) \left[|f'(a)| (A_2(a, b, 1, \varkappa) + A_5(a, b, 1, \varkappa)) + |f'(b)| (A_3(a, b, 1, \varkappa) + A_6(a, b, 1, \varkappa)) \right]. \end{aligned}$$

(ii) On letting $q \rightarrow 1^-$ and $\varkappa = \frac{\eta(a, b) + 2b}{2}$ in Theorem 3.1, we obtain the following midpoint inequality

$$\begin{aligned} & \left| \frac{1}{\eta(b, a)} \int_{b+\eta(a, b)}^b f(t) dt - f\left(\frac{\eta(a, b) + 2b}{2}\right) \right| \tag{14} \\ & \leq \frac{\eta(b, a)}{8} \left[\left(\frac{|f'(a)|^{p_1} + 2|f'(b)|^{p_1}}{3} \right)^{\frac{1}{p_1}} + \left(\frac{2|f'(a)|^{p_1} + |f'(b)|^{p_1}}{3} \right)^{\frac{1}{p_1}} \right] \end{aligned}$$

which was proved by Sarikaya et al. in [39, Theorem 8].

Moreover, by considering $p_1 = 1$ in the inequality (14), we obtain the following midpoint inequality:

$$\left| \frac{1}{\eta(b, a)} \int_{b+\eta(a, b)}^b f(t) dt - f\left(\frac{\eta(a, b) + 2b}{2}\right) \right| \leq \frac{\eta(b, a)}{8} \left[|f'(a)| + |f'(b)| \right]$$

which was established by Sarikaya et al. in [39, Theorem 5].

(iii) By choosing $\varkappa = \frac{a+qb}{[2]_q}$ and $\eta(b, a) = -\eta(a, b) = b - a$, we have the following midpoint inequality

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) {}^b d_q t - f\left(\frac{a+qb}{[2]_q}\right) \right| \tag{15} \\ & \leq (b-a) \left[\left(\frac{q}{[2]_q^3} \right)^{1-\frac{1}{p_1}} \right. \\ & \quad \times \left(\left| {}^b D_q f(a) \right|^{p_1} \left(\frac{q}{[2]_q^3 [3]_q} \right) + \left| {}^b D_q f(b) \right|^{p_1} \left(\frac{q^2 + q^3}{[2]_q^3 [3]_q} \right) \right)^{\frac{1}{p_1}} \\ & \quad + \left(\frac{q}{[2]_q^3} \right)^{1-\frac{1}{p_1}} \\ & \quad \times \left(\left| {}^b D_q f(a) \right|^{p_1} \left(\frac{2q}{[2]_q^3 [3]_q} \right) + \left| {}^b D_q f(b) \right|^{p_1} \left(\frac{-q + q^2 + q^3}{[2]_q^3 [3]_q} \right) \right)^{\frac{1}{p_1}} \left. \right] \end{aligned}$$

which was proved by Ali et al. in [2].

Specifically, by applying the limit as $q \rightarrow 1^-$ and taking $p_1 = 1$, we have the following midpoint inequality in [25, Theorem 2.2]

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \leq (b-a) \left[\frac{|f'(a)| + |f'(b)|}{8} \right].$$

(iv) By taking $p_1 = 1$ and $\eta(b, a) = -\eta(a, b) = b - a$ in Theorem 3.1, we have the following Ostrowski type inequality

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) {}^b d_q t - f(\varkappa) \right| \\ & \leq (b-a) \left[\left| {}^b D_q f(a) \right| [A_2(a, b, q, \varkappa) + A_5(a, b, q, \varkappa)] \right. \\ & \quad \left. + \left| {}^b D_q f(b) \right| [A_3(a, b, q, \varkappa) + A_6(a, b, q, \varkappa)] \right]. \end{aligned} \tag{16}$$

(v) By setting $p_1 = 1$, $\varkappa = \frac{a+qb}{[2]_q}$, and $\eta(b, a) = -\eta(a, b) = b - a$, we have the following midpoint type inequality in [12],

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) {}^b d_q t - f\left(\frac{a+qb}{[2]_q}\right) \right| \\ & \leq (b-a) \left[\left| {}^b D_q f(a) \right| \left(\frac{3q}{[2]_q^3 [3]_q} \right) + \left| {}^b D_q f(b) \right| \left(\frac{-q + 2q^2 + 2q^3}{[2]_q^3 [3]_q} \right) \right]. \end{aligned} \tag{17}$$

(vi) On considering $p_1 = 1$, $|{}^b D_q f(\varkappa)| \leq M$, $q \rightarrow 1^-$, and $\eta(b, a) = -\eta(a, b) = b - a$ in Theorem 3.1, we obtain inequality (1).

Remark 4.3. If we consider Theorem 3.2. Then

(i) On taking the limit $q \rightarrow 1^-$ in Theorem 3.2, we obtain the following Ostrowski type inequality

$$\begin{aligned} & \left| \frac{1}{\eta(b, a)} \int_{b+\eta(a, b)}^b f(t) dt - f(\varkappa) \right| \\ & \leq \eta(b, a) \left[\left(\frac{b-\varkappa}{\eta(b, a)} \right)^{1+\frac{1}{r_1}} \left(\frac{1}{r_1+1} \right)^{\frac{1}{r_1}} \right. \\ & \quad \times \left(|f'(a)|^{p_1} \frac{1}{2} \left(\frac{b-\varkappa}{\eta(b, a)} \right)^2 + |f'(b)|^{p_1} \left(\frac{b-\varkappa}{\eta(b, a)} - \frac{1}{2} \left(\frac{b-\varkappa}{\eta(b, a)} \right)^2 \right) \right]^{\frac{1}{p_1}} \\ & \quad + \left(\frac{1}{r_1+1} \left(\frac{b+\eta(a, b)-\varkappa}{\eta(b, a)} \right)^{r_1+1} \right)^{\frac{1}{r_1}} \\ & \quad \times \left(|f'(a)|^{p_1} \left(\frac{1}{2} \left(1 - \left(\frac{b-\varkappa}{\eta(b, a)} \right)^2 \right) \right) \right. \\ & \quad \left. + |f'(b)|^{p_1} \left(\frac{1}{2} - \frac{b-\varkappa}{\eta(b, a)} + \frac{1}{2} \left(\frac{b-\varkappa}{\eta(b, a)} \right)^2 \right) \right)^{\frac{1}{p_1}} \Big]. \end{aligned}$$

(ii) By letting $q \rightarrow 1^-$ and $\varkappa = \frac{\eta(a,b)+2b}{2}$ in Theorem 3.2, Theorem 3.2 reduces to [39, Theorem 6].

(iii) By using $|{}^bD_q f(\varkappa)| \leq M$ and $\eta(b, a) = -\eta(a, b) = b - a$ in Theorem 3.2, we have the following new quantum Ostrowski type inequality

$$\left| \frac{1}{b-a} \int_a^b f(t) {}^b d_q t - f(\varkappa) \right| \leq M(b-a) \left[\left(\frac{b-\varkappa}{b-a} \right)^2 \left(\frac{q}{[r_1+1]_q} \right)^{\frac{1}{r_1}} + \left(\int_{\frac{b-\varkappa}{b-a}}^1 (1-qt)^{r_1} d_q t \right)^{\frac{1}{r_1}} \left(\frac{\varkappa-a}{b-a} \right)^{\frac{1}{p_1}} \right]. \quad (18)$$

Specifically, by taking the limit as $q \rightarrow 1^-$ in (18), we obtain the following Ostrowski type inequality given in [4, Theorem 3 with $s=1$];

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f(\varkappa) \right| \leq \frac{M(b-a)}{(r_1+1)^{\frac{1}{r_1}}} \left[\left(\frac{\varkappa-a}{b-a} \right)^2 + \left(\frac{b-\varkappa}{b-a} \right)^2 \right].$$

(iv) By choosing $\varkappa = \frac{a+qb}{[2]_q}$ and $\eta(b, a) = -\eta(a, b) = b - a$ in Theorem 3.2, we have the following new midpoint inequality

$$\left| \frac{1}{b-a} \int_a^b f(t) {}^b d_q t - f\left(\frac{a+qb}{[2]_q}\right) \right| \leq (b-a) \left[\left(\frac{1}{[2]_q} \right)^{1+\frac{1}{r_1}} \left(\frac{q}{[r_1+1]_q} \right)^{\frac{1}{r_1}} \left(|{}^bD_q f(a)|^{p_1} \frac{1}{[2]_q^3} + |{}^bD_q f(b)|^{p_1} \frac{2q+q^2}{[2]_q^3} \right)^{\frac{1}{p_1}} + \left(\int_{\frac{1}{[2]_q}}^1 (1-qt)^{r_1} d_q t \right)^{\frac{1}{r_1}} \left(|{}^bD_q f(a)|^{p_1} \frac{2q+q^2}{[2]_q^3} + |{}^bD_q f(b)|^{p_1} \frac{q^3+q^2-q}{[2]_q^3} \right) \right]. \quad (19)$$

Specifically, by taking the limit as $q \rightarrow 1^-$ in (19), we obtain the following midpoint inequality

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4(r_1+1)^{\frac{1}{r_1}}} \left[\left(\frac{|f'(a)|^{p_1} + 3|f'(b)|^{p_1}}{4} \right)^{\frac{1}{p_1}} + \left(\frac{3|f'(a)|^{p_1} + |f'(b)|^{p_1}}{4} \right)^{\frac{1}{p_1}} \right] \\ & \leq \left(\frac{b-a}{4} \right) \left(\frac{4}{r_1+1} \right)^{\frac{1}{r_1}} [|f'(a)| + |f'(b)|] \end{aligned}$$

which was proved by Kırmacı in [25].

5. CONCLUSIONS

In this investigation, we have derived a generalized variant of the Montgomery identity employing the quantum integral. We have proved some new Ostrowski's type inequalities for q -differentiable preinvex functions by using the newly offered identity. We have discussed the special cases of the newly offered results and obtained several new and known Ostrowski's and midpoint inequalities. It is an interesting and new problem that other mathematicians can prove for different kinds of convexities in their future work.

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